CIRCULAR MOTION

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CIRCULAR MOTION

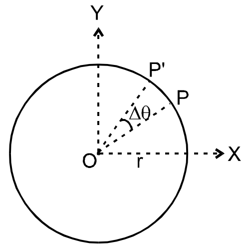
When a particle moves in a plane such that its distance from a fixed (or moving) point remains constant then its motion is called as the circular motion with respect to that fixed (or moving) point. That fixed point is called centre and the distance is called radius.

KINEMATICS OF CIRCULAR MOTION

Variables of Motion

(a) Angular Position

The angle made by the position vector with given line (reference line) is called angular position.



Circular motion is a two dimensional motion of motion in a plane.

Suppose a particle P is moving in a circle of radius r and centre O.

The position of the particle P at a given instant may be described by the angle θ between OP and OX. This angle θ is called the **angular position** of the particle. As the particle moves on the circle its angular position θ change. Suppose the point rotates an angle $\Delta\theta$ in time Δt.

(b) Angular Displacement

Definition: Angle rotated by a position vector of the moving particle with some reference line is called angular displacement.

Important points:

1. It is dimensionless and has proper unit (SI unit) radian while other units are degree or revolution

$$2\pi \text{ rad} = 360° = 1 \text{ rev}$$

2. Infinitely small angular displacement is a vector quantity but finite angular displacement is not because the addition of the small angular displacement is commutative while for large is not.

$$d\vec{\theta}_1 + d\vec{\theta}_2 = d\vec{\theta}_2 + d\vec{\theta}_1 \quad \text{but} \quad \theta_1 + \theta_2 \neq \theta_2 + \theta_1$$

3. Direction of small angular displacement is decided by right hand thumb rule. When the figures are directed along the motion of the point then thumb will represents the direction of angular displacement.

4. Angular displacement can be different for different observers

(c) Angular Velocity ω

(i) Average Angular Velocity

$$\omega_{av} = \frac{\text{Total Angle of Rotation}}{\text{Total time taken}} \; ; \quad \omega_{av} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

where θ_1 and θ_2 are angular position of the particle at time t_1 and t_2

(ii) Instantaneous Angular Velocity

The rate at which the position vector of a particle w.r.t. the centre rotates, is called as instantaneous angular velocity with respect to the centre.

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Important points:

1. It is an axial vector with dimensions $[T^{-1}]$ and SI unit rad/s.

2. For a rigid body as all points will rotate through same angle in same time, angular velocity is a characteristic of the body as a whole, e.g., angular velocity of all points of earth about its own axis is $(2\pi/24)$ rad/hr.

3. If a body makes 'n' rotations in 't' seconds then angular velocity in radian per second will be

$$\omega_{av} = \frac{2\pi n}{t}$$

If T is the period and 'f' the frequency of uniform circular motion $\omega_{av} = \frac{2\pi \times 1}{T} = 2\pi f$

4. If $\theta = a - bt + ct^2$ then $\omega = \frac{d\theta}{dt} = -b + 2ct$

Ex. 1 Is the angular velocity of rotation of hour hand of a watch greater of smaller than the angular velocity of Earth's rotation about its own axis.

Ans. Hourhand completes one rotation in 12 hours while Earth completes one rotation in 24 hours. So, angular velocity of hour hand is double the angular velocity of Earth. $\left(\omega = \frac{2\pi}{T}\right)$.

(d) Angular Acceleration α

(i) Average Angular Acceleration :

Let ω_1 and ω_2 be the instantaneous angular speeds at times t_1 and t_2 respectively, then the average angular acceleration α_{av} is defined as

$$\alpha_{av} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

(ii) Instantaneous Angular Acceleration :

It is the limit of average angular acceleration as Δt approaches zero, i.e.,

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \omega \frac{d\omega}{d\theta}$$

Important points:

1. It is also an axial vector with dimension $[T^{-2}]$ and unit rad/s^2.

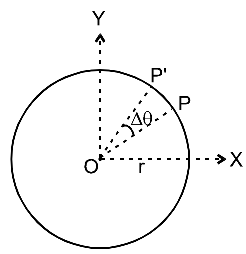
2. If $\alpha = 0$, circular motion is said to be uniform.

3. As $\omega = \frac{d\theta}{dt}, \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$,

i.e., second derivative of angular displacement w.r.t. time gives angular acceleration.

RELATION BETWEEN SPEED AND ANGULAR VELOCITY

$$\omega = \lim_{\Delta t \to \theta} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$



The rate of change of angular velocity is called the angular acceleration (α). Thus,

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

The linear distance PP' travelled by the particle in time Δt is

$$\Delta s = r\Delta\theta$$

or $$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = r \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

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or $$\frac{\Delta s}{\Delta t} = r\frac{d\theta}{dt}$$

or $$v = r\omega$$

Here, v is the linear speed of the particle.

Differentiating again with respect to time, we have

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} \text{ or } a_t = r\alpha$$

Here, $a_t = \frac{dv}{dt}$ is the rate of change of speed (not the rate of change of velocity). This is called the tangential acceleration of the particle. Later, we will see that a_t is the component of net acceleration $\vec{a}$ of the particle moving in a circle along the tangent.

Ex. 2 A particle travels in a circle of radius 20 cm at a speed that uniform increases. If the speed changes from 5.0 m/s to 6.0 m/s in 2.0s, find the angular acceleration.

Sol. The tangential acceleration is given by

$$a_1 = \frac{dv}{dt} = \frac{v_2 - v_1}{t_2 - t_1}$$

$$= \frac{6.0 - 5.0}{2.0} \text{ m/s}^2 = 0.5 \text{ m/s}^2.$$

The angular acceleration is $\alpha = a_t / r$

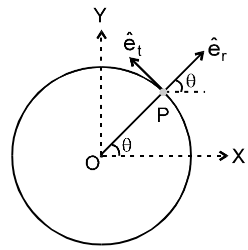
$$= \frac{0.5\,\text{m/s}^2}{20\,\text{cm}} = 2.5 \text{ rad/s}^2.$$

RADIAL AND TANGENTIAL ACCELERATION

Unit vectors along the radius and the tangent

Consider a particle P moving in a circle of radius r and centre at origin O. The angular position of the particle at some instant is say θ. Let us here define two unit vectors, one is $\hat{e}_r$ (called radial unit vector) which is along OP and the other is $\hat{e}_t$ (called the tangential unit vector) which is perpendicular to OP. Now, since

$$|\hat{e}_r| = |\hat{e}_t| = 1$$



We can write these two vectors as

$$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

and $$\hat{e}_t = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

Velocity and acceleration of particle in circular motion :

The position vector of particle P at the instant shown in figure can be written as

$$\vec{r} = \overrightarrow{OP} = r\hat{e}_r$$

or $$\vec{r} = r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

The velocity of the particle can be obtained by differentiating $\vec{r}$ with respect to time t. Thus,

$$\vec{v} = \frac{d\vec{r}}{dt} = (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})\,r\omega$$

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$= r\left[\omega\frac{d}{dt}(-\sin\theta\hat{i}+\cos\theta\hat{j})+(-\sin\theta\hat{i}+\cos\theta\hat{j})\frac{d\omega}{dt}\right]$$

$$= -\omega^2 r\,[\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}] + r\frac{d\omega}{dt}\,\hat{e}_t$$

$$\vec{\mathbf{a}} = -\omega^2 r\,\hat{\mathbf{e}}_\mathbf{r} + \frac{dv}{dt}\,\hat{e}_t$$

Thus, acceleration of a particle moving in a circle has two components one is along $\hat{e}_t$ (along tangent) and the other along $-\hat{\mathbf{e}}_\mathbf{r}$ (or towards centre). Of these the first one is called the tangential acceleration. (a_t) and the other is called radial or centripetal acceleration (a_r). Thus.

$$a_t = \frac{dv}{dt} = \text{rate of change of speed}$$

and $$a_r = rw^2 = r\left(\frac{v}{r}\right)^2 = \frac{v^2}{r}$$

Here, the two components are mutually perpendicular. Therefore, net acceleration of the particle will be :

$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{(r\omega^2)^2 + \left(\frac{dv}{dt}\right)^2} = \sqrt{\left(\frac{v^2}{r}\right)^2 + \left(\frac{dv}{dt}\right)^2}$$

Following three points are important regarding the above discussion:

1. In uniform circular motion, speed (v) of the particle is constant, i.e., $\frac{dv}{dt} = 0$. Thus,

$a_t = 0$ and $a = a_r = r\omega^2$

2. In accelerated circular motion, $\frac{dv}{dt}$ = positive, i.e., a_t is along $\hat{\mathbf{e}}_t$ or tangential acceleration of particle is parallel to velocity $\vec{v}$ because $\vec{v} = r\omega\,\hat{\mathbf{e}}_t$ and $\vec{a}_r = \frac{dv}{dt}\,\hat{\mathbf{e}}_t$

3. In decelerated circular motion, $\frac{dv}{dt}$ = negative and hence, tangential acceleration is anti-parallel to velocity $\vec{v}$.

Ex. 3 A particle moves in a circle of radius 2.0 cm at a speed given by v = 4t, where v is in cm/s and t in seconds.
(a) Find the tangential acceleration at t = 1s.
(b) Find total acceleration at t = 1s.

Sol. (a) Tangential acceleration

$$a_t = \frac{dv}{dt}$$

or $$a_t = \frac{d}{dt}(4t) = 4\ \text{cm/s}^2$$

$$a_C = \frac{v^2}{R} = \frac{(4)^2}{2} = 8 \qquad \Rightarrow \qquad a = \sqrt{a_t^2 + a_C^2} = \sqrt{(4)^2 + (8)^2} = 4\sqrt{5}\ \text{m/s}^2$$

Ex. 4 A boy whirls a stone in a horizontal circle of radius 1.5 m and at height 2.0 m above level ground. The string breaks, and the stone files off horizontally and strikes the ground after traveling a horizontal distance of 10 m.

What is the magnitude of the cetripetal acceleration of the stone while in circular motion?

Sol. $t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2\times 2}{9.8}} = 0.64\,s$

$v = \frac{10}{t} = 15.63$ m/s

$a = \frac{v_B^2}{R} = 0.45$ m/s^2

Ex. 5 Find the magnitude of the linear acceleration of a particle moving in a circle of radius 10 cm with uniform speed completing the circle in 4s.

Sol. The distance covered in completing the circle is $2\pi r = 2\pi \times 10$ cm. The linear speed is

$v = 2\pi r/t$

$= \frac{2\pi\times 10\,cm}{4s} = 5\pi$ cm/s.

The linear acceleration is

$a = \frac{v^2}{r} = \frac{(5\pi cm/s)^2}{10cm} = 2.5\pi^2$ cm/s^2.

Ex. 6 A particle moves in a circle of radius 20 am. Its linear speed is given by v = 2t where t is in second and v in meter/second . Find the radical and tangential acceleration at t = 3s.

Sol. The linear speed at t = 3s is

v = 2t = 6 m/s.

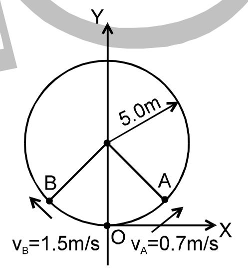
The radical acceleration at t = 3s is

$a_r = v^2/r = \frac{36 m^2/s^2}{0.20 m} = 180$ m/s^2.

The tangent acceleration is

$a_t = \frac{dv}{dt} = \frac{d(2t)}{dt} = 2$ m/s^2.

Ex. 7 Two particles A and B start at the origin O and travel in opposite directions along the circular path at constant speeds $v_A = 0.7$ m/s and $v_B = 1.5$ m/s, respectively. Determine the time when they collide and the magnitude of the acceleration of B just before this happens.



Sol. $1.5t + 0.7t = 2\pi R = 10\pi$

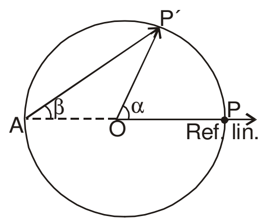
$\therefore$ $t = \frac{10\pi}{2.2} = 14.3$ s

$a = \frac{v_B^2}{R} = 0.45$ m/s^2

RELATIVE ANGULAR VELOCITY

Angular velocity is defined with respect to the point from which the position vector of the moving particle is drawn

Here angular velocity of the particle w.r.t. 'O' and 'A' will be different



$\omega_{PO} = \frac{d\alpha}{dt}$; $\omega_{PO} = \frac{d\beta}{dt}$

Definition:

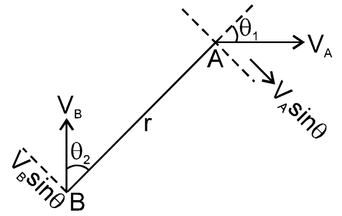
Relative angular velocity of a particle 'A' with respect to the other moving particle 'B' is the angular velocity of the position vector of 'A' with respect to 'B'. That means it is the rate at which position vector of 'A' with respect to 'B' rotates at that instant

$$\omega_{AB} = \frac{(V_{AB})\perp}{r_{AB}} = \frac{\text{Relative velocity of A w.r.t. B perpendicular to line AB}}{\text{Seperation between A and B}}$$

$(V_{AB})_{\perp} = V_A \sin\theta_1 + V_B \sin\theta_2$

$r_{AB} = r$

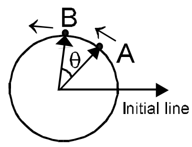
$$w_{AB} = \frac{v_B \sin\theta_1 + v_B \sin\theta_2}{r}$$

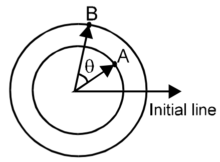


Important points:

1. If two particles are moving on the same circle or different coplanar concentric circles in same direction with different uniform angular speed ω_A and ω_B respectively, the angular velocity of B relative to A for an observer at the center will be

$$\omega_{BA} = \omega_B - \omega_A = \frac{d\theta}{dt}$$





So the time taken by one to complete one revolution around O w.r.t. the other

$$T = \frac{2\pi}{\omega_{rel}} = \frac{2\pi}{\omega_2 - \omega_1} = \frac{T_1 T_2}{T_1 - T_2}$$

2. If two particles are moving on two different concentric circles with different velocities then angular velocity of B relative to

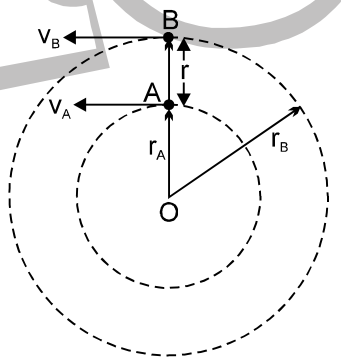
A as observed by A will depend on their positions and velocities. consider the case when A and B are closet to each other moving in same direction as shown in figure. In this situation

$v_{rel} = |\vec{v}_B - \vec{v}_A| = v_B - v_A$

$r_{rel} = |\vec{r}_B - \vec{r}_A| = r_B - r_A$

so, $\omega_{BA} = \frac{(v_{rel})_{\perp}}{r_{rel}} = \frac{v_B - v_A}{r_B - r_A}$

$(v_{rel})_{\perp}$ = Relative velocity perpendicular to position vector



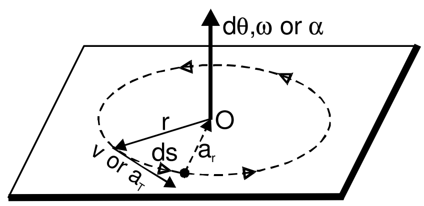
RELATIONS AMONG ANGULAR VARIABLES

These relations are also referred as equations of rotational motion and are –

$\omega = \omega_0 + \alpha t$ - (1)

$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ - (2)

$\omega^2 = \omega_0^2 + 2\alpha\theta$ - (3)



These are valid only if angular acceleration is constant and are analogous to equations of translatory motion, i.e.,

$v = u + at$; $s = ut + (1/2)\, at^2$ and $v^2 = u^2 + 2as$

RADIUS OF CURVATURE

Any curved path can be assume to be made of infinite circular arcs. Radius of curvature at a point is the radius of the circular arc at a particular point which fits the curve at that point.

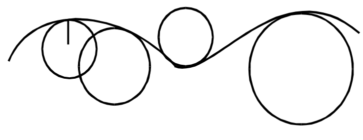
$$F_c = \frac{mv^2}{R} \Rightarrow R = \frac{mv^2}{F_C} = \frac{mv^2}{F_\perp}$$

$F_\perp$ = Force perpendicular to velocity (centripetal force)

If the equation of trajectory of a particle is given we can

find the radius of curvature of the instantaneous circle by using the formula ,

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}.$$



DYNAMICS OF CIRCULAR MOTION

In circular motion or motion along any curved path Newton's law is applied in two perpendicular directions one along the tangent and other perpendicular to it . i.e. towards centre. The component of net force along the centre is called **centripetal force**. The component of net force along the tangent is called **tangential force**.

$$\text{tangential force } (F_t) = Ma_t = M\frac{dv}{dt} = M\,\alpha\, r$$

$$\text{centripetal force } (F_c) = m\,\omega^2 r = \frac{mv^2}{r}$$

8. A small block of mass 100 g moves with uniform speed in a horizontal circular groove, with vertical side walls , of radius 25 cm. If the block takes 2.0s to complete one round, find the normal contact force by the slide wall of the groove.

Sol. The speed of the block is

$$v = \frac{2\pi \times (25\,cm)}{2.0\,s} = 0.785 \text{ m/s}$$

The acceleration of the block is

$$a = \frac{v^2}{r} = \frac{(0.785\,m/s)^2}{0.25} = 2.5 \text{ m/s}^2.$$

towards the center. The only force in this direction is the normal contact force due to the slide walls. Thus from Newton's second law , this force is

$$\mathcal{N} = ma = (0.100 \text{ kg})\,(2.5 \text{ m/s}^2) = 0.25 \text{ N}$$

CENTRIPETAL FORCE

Concepts : This is necessary resultant force towards the centre called the centripetal force.

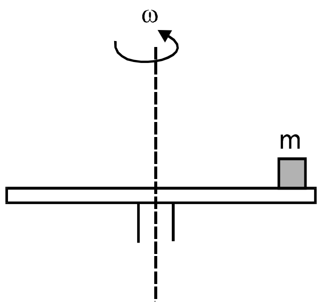
$$F = \frac{mv^2}{r} = m\omega^2 r$$

(i) A body moving with constant speed in a circle is not in equilibrium.

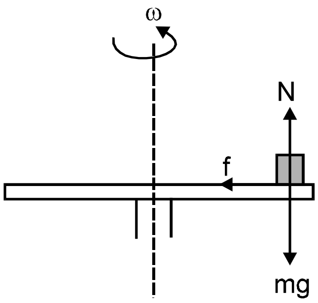
(ii) It should be remembered that in the absence of the centripetal force the body will move in a straight line with constant speed.

(iii) It is not a new kind of force which acts on bodies. In fact, any force which is directed towards the centre may provide the necessary centripetal force.

A small block of mass m, is at rest relative to turntable which rotates with constant angular speed ω.



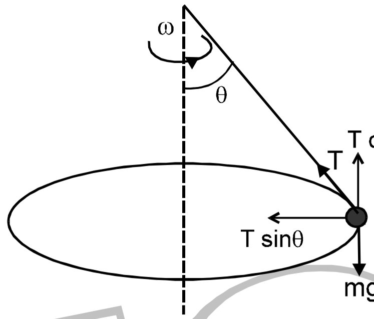
(A)
A block at rest with respect to a turn table



(B)
FBD of block with respect to inertial frame

CIRCULAR MOTION IN HORIZONTAL PLANE

A ball of mass m attached to a light and inextensible string rotates in a horizontal circle of radius r with an angular speed ω about the vertical. If we draw the force diagram of the ball. We can easily see that the component of tension force along the centre gives the centripetal force and component of tension along vertical balances the gravitation force.



FBD of ball w.r.t. ground

Ex. 9 An aircraft executes a horizontal loop of radius 1 km with a steady speed of 900 km h^{-1}. Compare its centripetal acceleration with the acceleration due to gravity.

Sol. $r = 1 \text{ km} = 10^3 \text{ m}$;

$$v = 900 \text{ km h}^{-1} = 900 \times \frac{5}{18} \text{ ms}^{-1} = 250 \text{ m s}^{-1}$$

Centripetal acceleration,

$$a_c = \frac{v^2}{r} = \frac{250 \times 250}{10^3} \text{ m s}^{-2} = 62.5 \text{ m s}^{-2}$$

Now, $\frac{a_c}{g} = \frac{62.5}{10} = 6.25$

Ex. 10 A stone tied to the end of a string 80 cm long is whirled in a horizontal circle with a constant speed. If the stone makes 14 revolutions in 25 s. What is the magnitude and direction of acceleration of the stone?

Sol. $r = 80 \text{ cm} = 0.80 \text{ m}; \ \omega = \frac{14 \text{ revolutions}}{25 \text{ s}} \times 2\pi$

Centripetal acceleration of $r\omega^2$

$$= 4 \times \frac{22}{7} \times \frac{22}{7} \times \frac{14}{25} \times \frac{14}{25} \times 0.8 = 9.9 \text{ ms}^{-2}$$

At every point, the acceleration is along the radius and towards the centre.

Ex. 11 A particle of mass m is suspended from a ceiling through a string of length L. The particle moves in a horizontal circle of radius r. Find (a) the speed of the particle and (b) the tension in the string. Such a system

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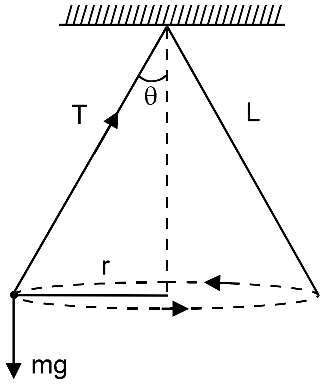
is called a conical pendulum .

Sol. The situation is shown in figure. The angle θ made by the string with the vertical is given by

$$\sin\theta = r / L \qquad ...(i)$$

The forces on the particle are
(a) the tension T along the string and
(b) the weight mg vertically downward.



The particle is moving in a circle with a constant speed v. Thus , the radical acceleration towards the centre has magnitude v^2 / r. Resolving the forces along the radial direction and applying . Newton's second law,

$$T\sin\theta = m(v^2 / r) \qquad ....(ii)$$

As there is no acceleration in vertical directions, we have from Newton's law,

$$T\cos\theta = mg \qquad .....(iii)$$

Dividing (ii) by (iii),

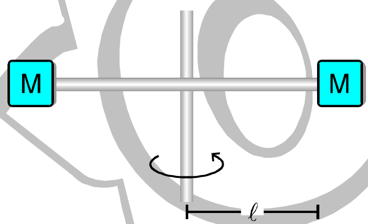
$$\tan\theta = \frac{v^2}{rg}$$

or, $$v = \sqrt{rg\tan\theta}$$

And from (iii), $$T = \frac{mg}{\cos\theta}$$

Using (i), $$v = \frac{r\sqrt{g}}{(L^2 - r^2)^{\frac{1}{4}}} \text{ and } \frac{mgL}{(L^2 - r^2)^{\frac{1}{2}}}.$$

Ex. 12 Two blocks each of mass M are connected to the ends of a light frame as shown in figure. The frame is rotated about the vertical line of symmetry. The rod brakes if the tension in it exceeds T_0. Find the maximum frequency with which the frame may be rotated without braking the rod.



Sol. Consider one of the blocks . If the frequency of revolution is f, the angular velocity is $\varpi = 2\pi f$. The acceleration towards the centre is $v^2 / l = \varpi^2 l = 4\pi^2 f^2 l$. The only horizontal force on the block is the tension of the rod. At the point of braking , this force is T_0. So from Newton's law,

$$T_0 = M.4\pi^2 f^2 l$$

or, $$f = \frac{1}{2\pi}\left[\frac{T_0}{M\ell}\right]^{1/2}$$

Q. 1 A particle of mass 14 g attached to a string of 70 cm length is whirled round in a horizontal circle. If the period of revolution is 2 second, calculate the tension.

Ans. 9680 dyne

Q. 2 A string breaks under a load of 50 kg. A mass of 1 kg is attached to one end of the string 10 m long and is rotated in horizontal circle. Calculate the greatest number of revolutions that the mass can make without breaking the string.

Ans. n = 1.114 revolutions per second.

MOTION IN A VERTICAL CIRCLE

To understand this consider the motion of a small body (say stone) tied to a string and whirled in a vertical circle. If at any time the body is at angular position θ, as shown in the figure, the forces acting on it are tension T in the string along the radius towards the center and the weight of the body mg acting vertically down wards.

Applying Newton's law towards centre

$$T - mg\cos\theta = \frac{mv^2}{r} \quad \text{or} \quad T = \frac{mv^2}{r} + mg\cos\theta$$

The body will move on the circular path only and only if $T_{min} > 0$ (as if $T_{min} \le 0$, the string will slack and the body will fall down instead of moving on the circle). So for completing the circle, i.e., 'looping the loop'

$$\frac{mv_H^2}{r} - mg \ge 0 \quad \text{i.e.,} \quad v_H \ge \sqrt{gr} \quad ...(1)$$

Now applying conservation of mechanical energy between highest point H and lowest point L

we get $\qquad v_L \ge \sqrt{5gr}$

i.e., for looping the loop, velocity at lowest point must be $\ge \sqrt{5gr}$.

In case of motion in a vertical plane tension is maximum at lowest position and in case of looping the loop

$T_{min} \ge 6mg$.

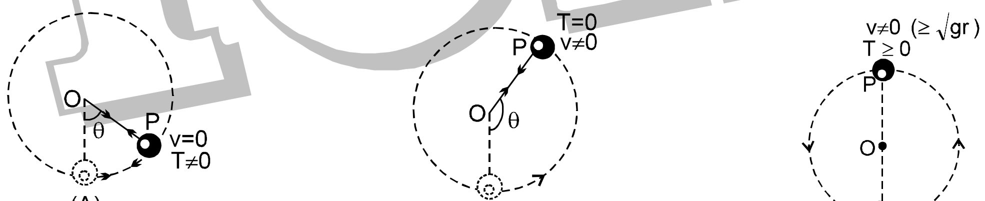
CONDITION FOR OSCILLATION OR LEAVING THE CIRCLE

In case of non uniform circular motion in a vertical plane if velocity of body at lowest point is lesser than $\sqrt{5gr}$, the particle will not complete the circle in vertical plane. Now it can either oscillate about the lowest point or after reaching a certain height may loose contact with the path.

From the theory of looping the loop we know that if $v_L \ge \sqrt{5gr}$, the body will loop the loop. So if the velocity of a body at lowest point is such that–

$$\sqrt{2gr} < v_L < \sqrt{5gr}$$

the body will move along the circle for $\theta > 90°$ and will not reach upto highest point but will leave the circle somewhere between $90° < \theta < 180°$. Here it is worth noting that at the point of leaving the circle T = 0 but $v_0 \neq 0$. This all is shown in figure.



(A) For Oscillation $0 < v_L \le \sqrt{2gr}$ $0 < \theta \le 90°$

(B) For Leaving the circle $\sqrt{2gr} < v_L < \sqrt{5gr}$ $90° < \theta < 180°$

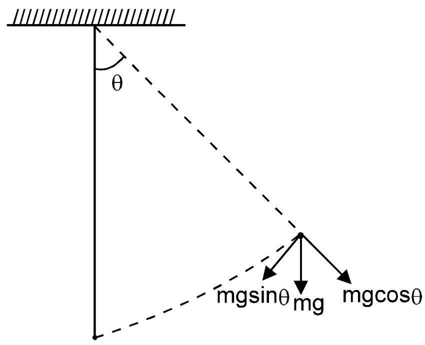
(C) For Looping the loop $v_L \ge \sqrt{5gr}$

Ex. 13 A simple pendulum is constructed by attaching a bob of mass m to a string of length L fixed at its upper end. The bob oscillates in a vertical circle. It is found that the speed of the bob is v when the string makes an angle θ with the vertical. Find the tension in the string at this instant.

Sol. The forces acting on the bob are (figure)
(a) the tension T
(b) the weight mg.
As the bob moves in a vertical circle with centre at O, the radial acceleration is v^2 / L towards O. Taking the components along this radius and applying Newton's second law , we get

$$T - mg\cos\theta = mv^2 / L$$

or, $\qquad T = m(g\cos + v^2 / L)$.



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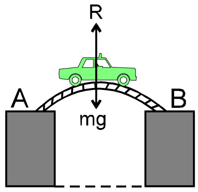
Ex. 14 Prove that a motor car moving over a convex bridge is lighter than the same car resting on the same bridge.

Sol. The motion of the motor car over a convex bridge AB is the motion along the segment of a circle AB (Figure;



The centripetal force is provided by the difference of weight mg of the car and the normal reaction R of the bridge.

$\therefore \quad mg - R = \frac{mv^2}{r} \quad$ or $\quad R = mg - \frac{mv^2}{r}$

Clearly R < mg, i.e., the weight of the moving car is less than the weight of the stationary car.

Ex. 15 A body weighing 0.4 kg is whirled in a vertical circle with a string making 2 revolutions per second. If the radius of the circle is 1.2 m. Find the tension (a) at the top of the circle, (b) at the bottom of the circle. Given : $g = 10\ m\ s^{-2}$ and $\pi = 3.14$.

Sol. Mass, m = 0.4 kg;

time period = $\frac{1}{2}$ second, radius, r = 1.2 m

Angular velocity, $\omega = \frac{2\pi}{1/2} = 4\pi\ rad\ s^{-1} = 12.56\ rad\ s^{-1}$.

(a) At the top of the circle,.

$T = \frac{mv^2}{r} - mg = mr\omega^2 - mg = m(r\omega^2 - g)$

$= 0.4\ (1.2 \times 12.56 \times 12.56 - 9.8)\ N = 71.2\ N$

(b) At the lowest point, $T = m(r\omega^2 + g) = 80\ N$

Ex. 16 You may have seen in a circus a motorcyclist driving in vertical loops inside a 'death wall' (a hollow spherical chamber with holes, so that the cyclist does not drop down when he is at the uppermost point, with no support from below. What is the minimum speed required at the uppermost position to perform a vertical loop if the radius of the chamber is 25 m?

Sol. When the motorcyclist is at the highest point of the death-well, the normal reaction R on the motorcyclist by the ceiling of the chamber acts downwards. His weight mg also act downwards. These two forces are balanced by the outward centrifugal force acting on him

$\therefore \quad R + mg = \frac{mv^2}{r}$

Here v is the speed of the motorcyclist and m is the mass of the motorcyclist (including the mass of the motor cycle). Because of the balancing of the forces, the motorcyclist does not fall down.
The minimum speed required to perform a vertical loop is given by equation (1) when R = 0.

$\therefore \quad mg = \frac{mv_{min}^2}{r}$ or $v_{min}^2 = gr$

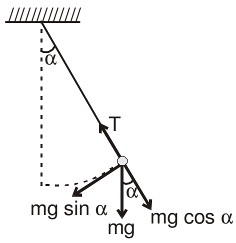
or $\quad v_{min} = \sqrt{gr} = \sqrt{9.8 \times 25}\ m\ s^{-1} = 15.65\ ms^{-1}$.

So, the minimum speed, at the top, required to perform a vertical loop is $15.65\ m\ s^{-1}$.

Ex. 17 A simple pendulum is constructed by attaching a bob of mass m to a string of length L fixed at its upper end. The bob oscillates in a vertical circle. It is found that the speed of the bob is v when the string makes an angle α with the vertical. Find the tension in the string and the magnitude of net force on the bob at the instant.

Sol. (i) The forces acting on the bob are :

(a) the tension T
(b) the weight mg



As the bob moves in a circle of radius L with centre at O. A centripetal force of magnitude $\frac{mv^2}{L}$ is required towards O. This force will be provided by the resultant of T and mg cos α. Thus,

or $\quad T - mg\cos\alpha = \frac{mv^2}{L} \qquad T = m\left(g\cos\alpha + \frac{v^2}{L}\right)$

(ii) $|\vec{F}_{net}| = \sqrt{(mg\ \sin\alpha)^2 + \left(\frac{mv^2}{L}\right)^2}$ $= m\sqrt{g^2 \sin^2\alpha + \frac{v^4}{L^2}}$

Q. 3 One end of a string of length 1.4 m is tied to a stone of mass 0.4 kg and the other end to a small pivot. Find the minimum velocity of stone required at its lowest point so that the string does not slacken at any point in its motion along the vertical circle?

Ans. 8.25 ms^{-1}

Q. 4 A particle of mass m slides without friction from the top of a hemisphere of radius r. At what height will the body lose contact with the surface of the sphere?

Ans. At a height of 2r/3 above the centre of the hemisphere.

CIRCULAR TURNING ON ROADS

When vehicles go through turnings, they travel along a nearly circular arc. There must be some force which will produce the required centripetal acceleration. If the vehicles travel in a horizontal circular path, this resultant force is also horizontal. The necessary centripetal force is being provided to the vehicles by following three ways.

1. **By friction only**
2. **By banking of roads only.**
3. **By friction and banking of roads both.**

In real life the necessary centripetal force is provided by friction and banking of roads both. Now let us write equations of motion in each of the three cases separately and see what are the constant in each case.

1. BY FRICTION ONLY

Suppose a car of mass m is moving at a speed v in a horizontal circular arc of radius r. In this case, the necessary centripetal force to the car will be provided by force of friction f acting towards center

Thus, $f = \frac{mv^2}{r}$

Further, limiting value of f is μN

or $f_L = \mu N = \mu mg \quad (N = mg)$

Therefore, for a safe turn without sliding $\frac{mv^2}{r} \le f_L$

or $\frac{mv^2}{r} \le \mu mg$ or $\mu \ge \frac{v^2}{rg}$ or $v \le \sqrt{\mu rg}$

Here, two situations may arise. If μ and r are known to us, the speed of the vehicle should not exceed $\sqrt{\mu rg}$ and if v and r are known to us, the coefficient of friction should be greater than $\frac{v^2}{rg}$.

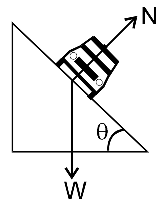
Q. 5 A bend in a level road has a radius of 100 m. Calculate the maximum speed which a car turning this bend may have without skidding. Given : $\mu = 0.8$.

Ans. 28 ms^{-1}

2. BY BANKING OF ROADS ONLY

Friction is not always reliable at circular turns if high speeds and sharp turns are involved. to avoid dependence on friction, the roads are banked at the turn so that the outer part of the road is some what lifted compared to the inner part.

Applying Newton's second law along the radius and the first law in the vertical direction.



$N \sin\theta = \frac{mv^2}{r}$ or $N \cos\theta = mg$

from these two equations, we get $\tan\theta = \frac{v^2}{rg}$ or $v = \sqrt{rg\tan\theta}$

Ex. 18 A circular track of radius 600 m is to be designed for cars at an average speed of 180 km/hr. What should be the angle of banking of the rack?

Sol. Let the angle of banking be θ. The forces on the car are (figure)

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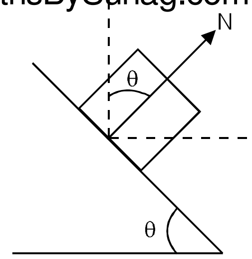
(a) weight of the car Mg downward and
(b) normal force N.
For proper banking , static frictional force is not needed.
For vertical direction the acceleration is zero. So,

$$N\cos\theta = Mg \qquad .....(i)$$

For horizontal direction , the acceleration is v^2/r towards the centre , so that

$$N\sin\theta = Mv^2/r \qquad .....(ii)$$



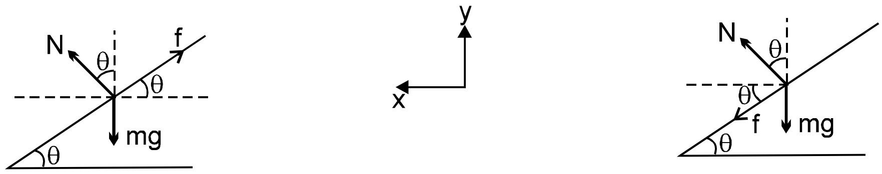
From (i) and (ii), $\tan\theta = v^2/rg$

Putting the values , $\tan\theta = \frac{180(km/hr)^2}{(600m)(10m/s^2)} = 0.4167$

$\theta = 22.6^\circ$.

3. BY FRICTION AND BANKING OF ROAD BOTH

If a vehicle is moving on a circular road which is rough and banked also, then three forces may act on the vehicle, of these the first force, i.e., weight (mg) is fixed both in magnitude and direction.



The direction of second force, i.e., normal reaction N is also fixed (perpendicular ro road) while the direction of the third force i.e., friction f can be either inwards or outwards while its magnitude can be varied upto a maximum limit ($f_L = \mu N$). So the magnitude of normal reaction N and directions plus magnitude of friction f are so adjusted that the resultant of the three forces mentioned above is $\frac{mv^2}{r}$ towards the center. Of these m and r are also constant. Therefore, magnitude of N and directions plus magnitude of friction mainly depends on the speed of the vehicle v. Thus, situation varies from problem to problem. Even though we can see that :

(i) Friction f will be outwards if the vehicle is at rest v = 0. Because in that case the component weight mg sinθ is balanced by f.
(ii) Friction f will be inwards if $v > \sqrt{rg\tan\theta}$
(iii) Friction f will be outwards if $v < \sqrt{rg\tan\theta}$ and
(iv) Friction f will be zero if $v = \sqrt{rg\tan\theta}$

NOTE : (i) The expression $\tan\theta = \frac{v^2}{rg}$ also gives the angle of banking for an aircraft, i.e., the angle through which it should tilt while negotiating a curve, to avoid deviation from the circular path.

(ii) The expression $\tan\theta = \frac{v^2}{rg}$ also gives the angle at which a cyclist should lean inward, when rounding a corner. In this case, θ is the angle which the cyclist must make with the vertical.

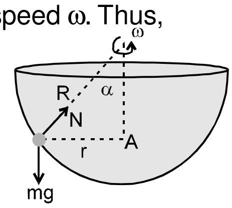
Ex. 19 A hemispherical bowl of radius R is rotating about its axis of symmetry which is kept vertical. A small ball kept in the bowl rotates with the bowl without slipping on its surface. If the surface of the bowl is smooth and the angle made by the radius through the ball with the vertical is α. Find the angular speed at which the bowl is rotating.

Sol. Let ω be the angular speed of rotation of the bowl. Two force are acting on the ball.
1. normal reaction N 2. weight mg

The ball is rotating in a circle of radius r (= R sin α) with centre at A at an angular speed ω. Thus,

$N\sin\alpha = mr\omega^2 = mRw^2\sin\alpha$ ...(i)

and $N\cos\alpha = mg$ ...(ii)



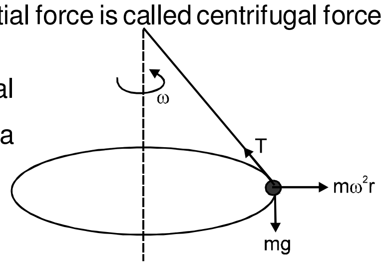
Dividing Eqs. (i) by (ii), we get $\frac{1}{\cos\alpha} = \frac{\omega^2 R}{g}$ $\therefore$ $\boldsymbol{\omega} = \sqrt{\frac{\mathbf{g}}{\mathbf{R\cos\alpha}}}$

CENTRIFUGAL FORCE

When a body is rotating in a circular path and the centripetal force vanishes, the body would leave the circular path. To an observer A who is not sharing the motion along the circular path, the body appears to fly off tangentially at the point of release. To another observer B, who is sharing the motion along the circular path (i.e., the observer B is also rotating with the body which is released, it appears to B, as if it has been thrown off along the radius away from the centre by some force. This inertial force is called centrifugal force.)

Its magnitude is equal to that of the centripetal force. = $\frac{mv^2}{r}$. Centrifugal force is a fictitious force which has to be applied as a concept only in a rotating frame of reference to apply N.L in that frame)

FBD of ball w.r.t. non inertial frame rotating with the ball.

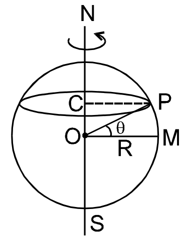


Suppose we are working from a frame of reference that is rotating at a constant, angular velocity ω with respect to an inertial frame. If we analyse the dynamics of a particle of mass m kept at a distance r from the axis of rotation, we have to assume that a force $mr\omega^2$ react radially outward on the particle. Only then we can apply Newton's laws of motion in the rotating frame. This radially outward pseudo force is called the centrifugal force.

EFFECT OF EARTHS ROTATION ON APPARENT WEIGHT

The earth rotates about its axis at an angular speed of one revolution per 24 hours. The line joining the north and the south poles is the axis of rotation.

Every point on the earth moves in a circle. A point at equator moves in a circle of radius equal to the radius of the earth and the centre of the circle is same as the centre of the earth. For any other point on the earth, the circle of rotation is smaller than this. Consider a place P on the earth (figure).



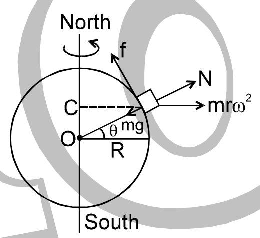
Drop a perpendicular PC from P to the axis SN. The place P rotates in a circle with the centre at C. The radius of this circle is CP. The angle between the line OM and the radius OP through P is called the latitude of the place P. We have

$$CP = OP\cos\theta \qquad \text{or,} \qquad r = R\cos\theta$$

where R is the radius of the earth.

If we work from the frame of reference of the earth, we shall have to assume the existence of pseudo force. In particular, a centrifugal force mw^2r has to be assumed on any particle of mass m placed at P.

If we consider a block of mass m at point P then this block is at rest with respect to earth. If resolve the forces along and perpendicular the centre of earth then



$$N + mr\omega^2\cos\theta = mg \quad \Rightarrow \quad N = mg - mr\omega^2\cos\theta \quad \Rightarrow \quad N = mg - mR\omega^2\cos^2\theta$$

Ex. 20 A body weighs 98N on a spring balance at the north pole. What will be its weight recorded on the same scale if it is shifted to the equator? Use $g = GM/R^2 = 9.8\ m/s^2$ and the radius of the earth R=6400 km.

Sol. At poles , the apparent weight is same as the true weight.

Thus, $98N = mg = m(9.8\ m/s^2)$

At the equator , the apparent weight is $mg' = mg - m\varpi^2 R$

The radius of the earth is 6400 km and the angular speed is $\varpi = \frac{2\pi\ rad}{24\times60\times60s} = 7.27\times10^{-6}$ rad/s

$mg' = 98N - (10\ kg)\ (7.27\times10^{-5}\ s^{-1})^2\ (6400\ km) \qquad = 97.66N$

